

Abstract

Considering the spin-coefficient version of the left-flat vacuum Einstein equations, all but one of the fifty equations can be explicitly integrated via the introduction of five spin-weight $s=-2$ complex potentials. The final equation is a non-linear wave equation for the last of the potentials. Solutions to this equation determine solutions for the entire system.

Solutions for several special cases are obtained.

1 I. Introduction

1.1 Background

The self-dual/left flat (or anti-self dual/right/flat) Ricci-flat equations and their spaces with associated metrics has been a subject of considerable interest since the late 1970s. The very large literature has been associated with a variety of linked interests.

They play a fundamental role in the properties of real asymptotically flat space-times via solutions to the 'good-cut' equation^{1,2,3,4,5,6,7}, they define Penrose's asymptotic twistor space^{8,9,10}, they (the solutions) are intimately related to (in fact are) Penrose's non-linear graviton fields, the equations themselves are studied as a beautiful example of a non-linear integrable system^{11,12,13} and finally the Euclidean versions, the so-called gravitational instantons^{14,15} (the general relativistic generalization of the Yang-Mills instantons) have played a role in attempts at understanding quantum gravity.

From the beginning there have been a variety of attempts at integrating the field equations and even now - 40 years after their popularity began - only a relatively small number of solutions appear to be known. The first solutions - a very limited number - were obtained via the 'good cut' equation. The best known is the Sparling-Tod metric⁶/ Eguchi-Hansen⁷ metric. Many others were discovered by imposing special conditions as for example algebraic specialness¹⁶ or (Killing) symmetries¹⁷. In addition there is a large literature on self-dual gravitational instantons¹⁴.

To our knowledge there have been very few attacks on the problem of finding (or even of studying) the general solution of the self-dual Einstein equations on general complex four dimensional manifolds - either globally or locally⁴. It is the purpose of this note to return to this issue.

Although all H-spaces (i.e., spaces arising from solutions of the good-cut equation) are self-dual spaces, it is not yet clear whether all self-dual spaces are H-spaces. Nevertheless we will not make a distinction here between the two.

1.2 Modus Operandi

We begin, in Sec.II, with the fully general Einstein equations in the spin-coefficient formalism written as complex equations on a thickened region of C^4 in a neighborhood of the real 'slice' R^4 . In the standard "real" version of the formalism, the basic field variables, i.e., the spin-coefficients, the metric

variables and Weyl tensor components, are, in general, complex. In the field equations themselves, their complex conjugates appear symmetrically so that the equations themselves, as a complete set are real. In our present (SD) version however, the previous "complex conjugates" are now completely freed up from their conjugate counterparts and are independent variables. These field equations are then reduced by setting the anti-self-dual part of the Weyl tensor to zero, thereby imposing the self-dual condition on the equations.

A coordinate system is introduced, in Sec.II, by choosing a world-line, \mathfrak{L} , in the thickened region of C^4 : with the (complex) u , being the 'complex time' along the world line. The null cones with apex on the line are labeled by u . The complex sphere of null directions at the apex of each cone (and their associated null geodesics) are labeled by the complex stereographic coordinates $(\zeta, \tilde{\zeta})$ while the affine parameter along the null geodesics is given by r . Note that both u and r have values close to the real while $\tilde{\zeta}$ is close to the complex conjugate $\bar{\zeta}$.

From these imposed conditions we easily show, Sec.II, that the complex divergence, ρ , of the null geodesics with apex on \mathfrak{L} is given by $\rho = -r^{-1}$ and the left shear by $\sigma = 0$.

At this point the entire set of field equations (about 50) are divided into three sets: (1) the radial equations which involve the r derivative, (2) those involving the angular derivatives, i.e., $(\zeta, \tilde{\zeta})$ and (3) those that contain the time-derivative, u - about(14) of them.

1.3 Results

What is a bit surprising is that every one of the r -derivative equations can be exactly integrated in terms of four spin-weight ($s = -2$), potentials, $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$, where each can be expressed as r -derivatives of the next one, so that the only independent one is γ_4 .

The angular equations then are then used to establish relationships between some of the free "constants" of the r -integrations.

Finally when all these results are inserted into the (14) time-derivative equations we discover that they are all closely related and can be reduced to a *single complex spin-weight* ($s = -2$), *non-linear wave equation* for the last potential γ_4 . This last equation carries all the evolution information for the entire set of equations.

Though we do not know of a way to give general solutions to this equation, it is quite easy to produce many special solutions.

2 II. The Field Equations

2.1 Spin-Coefficient version of Einstein equations

The familiar spin-coefficient version of the Einstein equations¹⁸- with their "conjugates" explicitly inserted and written out in detail - are:

The metric equations:

$$\begin{aligned}
\Delta l^a - Dn^a &= (\gamma + \bar{\gamma})l^a + (\epsilon + \bar{\epsilon})n^a - (\tau + \bar{\pi})\bar{m}^a - (\bar{\tau} + \pi)m^a, \\
\delta l^a - Dm^a &= (\bar{\alpha} + \beta - \bar{\pi})l^a + \kappa n^a - \sigma\bar{m}^a - (\bar{\rho} + \epsilon - \bar{\epsilon})m^a, \\
\bar{\delta}l^a - D\bar{m}^a &= (\alpha + \bar{\beta} - \pi)l^a + \bar{\kappa}n^a - \bar{\sigma}m^a - (\rho + \bar{\epsilon} - \epsilon)\bar{m}^a, \\
\delta n^a - \Delta m^a &= -\nu l^a + (\tau - \bar{\alpha} - \beta)n^a + \bar{\lambda}\bar{m}^a + (\mu - \gamma + \bar{\gamma})m^a, \\
\bar{\delta}n^a - \Delta\bar{m}^a &= -\bar{\nu}l^a + (\bar{\tau} - \alpha - \bar{\beta})n^a + \lambda m^a + (\bar{\mu} - \bar{\gamma} + \gamma)\bar{m}^a, \\
\bar{\delta}m^a - \delta\bar{m}^a &= (\bar{\mu} - \mu)l^a + (\bar{\rho} - \rho)n^a - (\bar{\alpha} - \beta)\bar{m}^a + (\alpha - \bar{\beta})m^a.
\end{aligned} \tag{1}$$

The spin-coefficient equations:

$$\begin{aligned}
\Delta\lambda - \bar{\delta}\nu &= -(\mu + \bar{\mu} + 3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4 \\
\Delta\bar{\lambda} - \delta\bar{\nu} &= -(\mu + \bar{\mu} + 3\bar{\gamma} - \gamma)\bar{\lambda} + (3\bar{\alpha} + \beta + \bar{\pi} - \tau)\bar{\nu} - \bar{\Psi}_4 \\
\delta\rho - \bar{\delta}\sigma &= \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 \\
\bar{\delta}\bar{\rho} - \delta\bar{\sigma} &= \bar{\rho}(\alpha + \bar{\beta}) - \bar{\sigma}(3\bar{\alpha} - \beta) + (\bar{\rho} - \rho)\bar{\tau} + (\bar{\mu} - \mu)\bar{\kappa} - \bar{\Psi}_1 \\
\delta\alpha - \bar{\delta}\beta &= \mu\rho - \lambda\sigma + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \epsilon(\mu - \bar{\mu}) - \Psi_2 \\
\bar{\delta}\bar{\alpha} - \delta\bar{\beta} &= \bar{\mu}\bar{\rho} - \bar{\lambda}\bar{\sigma} + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\bar{\alpha}\bar{\beta} - \bar{\gamma}(\rho - \bar{\rho}) - \bar{\epsilon}(\mu - \bar{\mu}) - \bar{\Psi}_2 \\
\delta\lambda - \bar{\delta}\mu &= (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\beta) - \Psi_3 \\
\bar{\delta}\bar{\lambda} - \delta\bar{\mu} &= -(\rho - \bar{\rho})\bar{\nu} - (\mu - \bar{\mu})\bar{\pi} + \bar{\mu}(\bar{\alpha} + \beta) + \bar{\lambda}(\alpha - 3\bar{\beta}) - \bar{\Psi}_3 \\
\delta\nu - \Delta\mu &= \mu^2 + \lambda\bar{\lambda} + \mu(\gamma + \bar{\gamma}) - \bar{\nu}\pi + \nu(\tau - 3\beta - \bar{\alpha}) \\
\bar{\delta}\bar{\nu} - \Delta\bar{\mu} &= \bar{\mu}^2 + \lambda\bar{\lambda} + \bar{\mu}(\gamma + \bar{\gamma}) - \nu\bar{\pi} + \bar{\nu}(\bar{\tau} - 3\bar{\beta} - \alpha) \\
\delta\gamma - \Delta\beta &= \gamma(\tau - \bar{\alpha} - \beta) + \mu\tau - \sigma\nu - \epsilon\bar{\nu} - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\bar{\lambda} \\
\bar{\delta}\bar{\gamma} - \Delta\bar{\beta} &= \bar{\gamma}(\bar{\tau} - \alpha - \bar{\beta}) + \bar{\mu}\bar{\tau} - \bar{\sigma}\bar{\nu} - \bar{\epsilon}\nu - \bar{\beta}(\bar{\gamma} - \gamma - \bar{\mu}) + \bar{\alpha}\lambda \\
\delta\tau - \Delta\sigma &= \mu\sigma + \rho\bar{\lambda} + \tau(\tau + \beta - \bar{\alpha}) - \sigma(3\gamma - \bar{\gamma}) - \kappa\bar{\nu} \\
\bar{\delta}\bar{\tau} - \Delta\bar{\sigma} &= \bar{\mu}\bar{\sigma} + \bar{\rho}\lambda + \bar{\tau}(\bar{\tau} + \bar{\beta} - \alpha) - \bar{\sigma}(3\bar{\gamma} - \gamma) - \bar{\kappa}\nu \\
\Delta\rho - \bar{\delta}\tau &= -\rho\bar{\mu} - \sigma\lambda + \tau(\bar{\beta} - \alpha - \bar{\tau}) + (\gamma + \bar{\gamma})\rho + \kappa\nu - \Psi_2 \\
\Delta\bar{\rho} - \delta\bar{\tau} &= -\bar{\rho}\mu - \bar{\sigma}\bar{\lambda} + \bar{\tau}(\beta - \bar{\alpha} - \tau) + (\gamma + \bar{\gamma})\bar{\rho} + \bar{\kappa}\nu - \bar{\Psi}_2 \\
\Delta\alpha - \bar{\delta}\gamma &= \nu(\rho + \epsilon) - \lambda(\tau + \beta) + \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) - \Psi_3 \\
\Delta\bar{\alpha} - \delta\bar{\gamma} &= \bar{\nu}(\bar{\rho} + \bar{\epsilon}) - \bar{\lambda}(\bar{\tau} + \bar{\beta}) + \bar{\alpha}(\gamma - \mu) + \bar{\gamma}(\beta - \tau) - \bar{\Psi}_3
\end{aligned} \tag{2}$$

$$\begin{aligned}
D\rho - \bar{\delta}\kappa &= \rho^2 + \sigma\bar{\sigma} + (\epsilon + \bar{\epsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) \\
D\bar{\rho} - \delta\bar{\kappa} &= \bar{\rho}^2 + \sigma\bar{\sigma} + (\epsilon + \bar{\epsilon})\bar{\rho} - \kappa\bar{\tau} - \bar{\kappa}(3\bar{\alpha} + \beta - \bar{\pi}) \\
D\sigma - \delta\kappa &= (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \\
D\bar{\sigma} - \bar{\delta}\bar{\kappa} &= (\rho + \bar{\rho})\bar{\sigma} + (3\bar{\epsilon} - \epsilon)\bar{\sigma} - (\bar{\tau} - \pi + \alpha + 3\bar{\beta})\bar{\kappa} + \bar{\Psi}_0
\end{aligned} \tag{3}$$

$$\begin{aligned}
D\tau - \Delta\kappa &= (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 \\
D\bar{\tau} - \Delta\bar{\kappa} &= (\bar{\tau} + \pi)\bar{\rho} + (\tau + \bar{\pi})\bar{\sigma} - (\epsilon - \bar{\epsilon})\bar{\tau} - (3\bar{\gamma} + \gamma)\bar{\kappa} + \bar{\Psi}_1
\end{aligned} \tag{4}$$

$$\begin{aligned}
D\alpha - \bar{\delta}\epsilon &= (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + (\epsilon + \rho)\pi \\
D\bar{\alpha} - \delta\bar{\epsilon} &= (\bar{\rho} + \epsilon - 2\bar{\epsilon})\bar{\alpha} + \bar{\beta}\sigma - \beta\bar{\epsilon} - \bar{\kappa}\bar{\lambda} - \kappa\bar{\gamma} + (\bar{\epsilon} + \bar{\rho})\bar{\pi}
\end{aligned} \tag{5}$$

$$\begin{aligned}
D\beta - \delta\epsilon &= (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1 \\
D\bar{\beta} - \bar{\delta}\bar{\epsilon} &= (\bar{\alpha} + \bar{\pi})\bar{\sigma} + (\rho - \epsilon)\bar{\beta} - (\bar{\mu} + \bar{\gamma})\bar{\kappa} - (\alpha - \pi)\bar{\epsilon} + \bar{\Psi}_1
\end{aligned} \tag{6}$$

$$\begin{aligned}
D\gamma - \Delta\epsilon &= (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\epsilon + \bar{\epsilon})\gamma - (\gamma + \bar{\gamma})\epsilon + \tau\pi - \nu\kappa + \Psi_2 \\
D\bar{\gamma} - \Delta\bar{\epsilon} &= (\bar{\tau} + \pi)\bar{\alpha} + (\tau + \bar{\pi})\bar{\beta} - (\epsilon + \bar{\epsilon})\bar{\gamma} - (\gamma + \bar{\gamma})\bar{\epsilon} + \bar{\tau}\bar{\pi} - \bar{\nu}\bar{\kappa} + \bar{\Psi}_2
\end{aligned} \tag{7}$$

$$\begin{aligned}
D\lambda - \bar{\delta}\pi &= \rho\lambda + \bar{\sigma}\mu + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} - (3\epsilon - \bar{\epsilon})\lambda \\
D\bar{\lambda} - \delta\bar{\pi} &= \bar{\rho}\bar{\lambda} + \sigma\bar{\mu} + \bar{\pi}^2 + (\bar{\alpha} - \beta)\bar{\pi} - \bar{\nu}\kappa - (3\bar{\epsilon} - \epsilon)\bar{\lambda}
\end{aligned} \tag{8}$$

$$\begin{aligned}
D\mu - \delta\pi &= \bar{\rho}\mu + \sigma\lambda + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\mu - \pi(\bar{\alpha} - \beta) - \nu\kappa + \Psi_2 \\
D\bar{\mu} - \bar{\delta}\bar{\pi} &= \rho\bar{\mu} + \bar{\sigma}\bar{\lambda} + \pi\bar{\pi} - (\epsilon + \bar{\epsilon})\bar{\mu} - \bar{\pi}(\alpha - \bar{\beta}) - \bar{\nu}\bar{\kappa} + \bar{\Psi}_2
\end{aligned} \tag{9}$$

$$\begin{aligned}
D\nu - \Delta\pi &= (\bar{\tau} + \pi)\mu + (\tau + \bar{\pi})\lambda + (\gamma - \bar{\gamma})\pi - (3\epsilon + \bar{\epsilon})\nu + \Psi_3 \\
D\bar{\nu} - \Delta\bar{\pi} &= (\tau + \bar{\pi})\bar{\mu} + (\bar{\tau} + \pi)\bar{\lambda} - (\gamma - \bar{\gamma})\bar{\pi} - (3\bar{\epsilon} + \epsilon)\bar{\nu} + \bar{\Psi}_3
\end{aligned} \tag{10}$$

and finally the Bianchi Identities:

$$\begin{aligned}
\bar{\delta}\Psi_0 - D\Psi_1 &= (4\alpha - \pi)\Psi_0 - 2(2\rho + \epsilon)\Psi_1 + 3\kappa\Psi_2, \\
\bar{\delta}\Psi_1 - D\Psi_2 &= \lambda\Psi_0 + 2(\alpha - \pi)\Psi_1 - 3\rho\Psi_2 + 2\kappa\Psi_3, \\
\bar{\delta}\Psi_2 - D\Psi_3 &= 2\lambda\Psi_1 - 3\pi\Psi_2 + 2(\epsilon - \rho)\Psi_3 + \kappa\Psi_4, \\
\bar{\delta}\Psi_3 - D\Psi_4 &= 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3 + (4\epsilon - \rho)\Psi_4,
\end{aligned} \tag{11}$$

$$\begin{aligned}
\delta\bar{\Psi}_0 - D\bar{\Psi}_1 &= (4\bar{\alpha} - \bar{\pi})\bar{\Psi}_0 - 2(2\bar{\rho} + \bar{\epsilon})\bar{\Psi}_1 + 3\bar{\kappa}\bar{\Psi}_2, \\
\delta\bar{\Psi}_1 - D\bar{\Psi}_2 &= \bar{\lambda}\bar{\Psi}_0 + 2(\bar{\alpha} - \bar{\pi})\bar{\Psi}_1 - 3\bar{\rho}\bar{\Psi}_2 + 2\bar{\kappa}\bar{\Psi}_3, \\
\delta\bar{\Psi}_2 - D\bar{\Psi}_3 &= 2\bar{\lambda}\bar{\Psi}_1 - 3\bar{\pi}\bar{\Psi}_2 + 2(\bar{\epsilon} - \bar{\rho})\bar{\Psi}_3 + \bar{\kappa}\bar{\Psi}_4, \\
\delta\bar{\Psi}_3 - D\bar{\Psi}_4 &= 3\bar{\lambda}\bar{\Psi}_2 - 2(\bar{\alpha} + 2\bar{\pi})\bar{\Psi}_3 + (4\bar{\epsilon} - \bar{\rho})\bar{\Psi}_4,
\end{aligned} \tag{12}$$

$$\begin{aligned}
\Delta\Psi_0 - \delta\Psi_1 &= (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2, \\
\Delta\Psi_1 - \delta\Psi_2 &= \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3, \\
\Delta\Psi_2 - \delta\Psi_3 &= 2\nu\Psi_1 - 3\mu\Psi_2 + 2(\beta - \tau)\Psi_3 + \sigma\Psi_4, \\
\Delta\Psi_3 - \delta\Psi_4 &= 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4.
\end{aligned} \tag{13}$$

$$\begin{aligned}
\Delta\bar{\Psi}_0 - \bar{\delta}\bar{\Psi}_1 &= (4\bar{\gamma} - \bar{\mu})\bar{\Psi}_0 - 2(2\bar{\tau} + \bar{\beta})\bar{\Psi}_1 + 3\bar{\sigma}\bar{\Psi}_2, \\
\Delta\bar{\Psi}_1 - \bar{\delta}\bar{\Psi}_2 &= \bar{\nu}\bar{\Psi}_0 + 2(\bar{\gamma} - \bar{\mu})\bar{\Psi}_1 - 3\bar{\tau}\bar{\Psi}_2 + 2\bar{\sigma}\bar{\Psi}_3, \\
\Delta\bar{\Psi}_2 - \bar{\delta}\bar{\Psi}_3 &= 2\bar{\nu}\bar{\Psi}_1 - 3\bar{\mu}\bar{\Psi}_2 + 2(\bar{\beta} - \bar{\tau})\bar{\Psi}_3 + \bar{\sigma}\bar{\Psi}_4, \\
\Delta\bar{\Psi}_3 - \bar{\delta}\bar{\Psi}_4 &= 3\bar{\nu}\bar{\Psi}_2 - 2(\bar{\gamma} + 2\bar{\mu})\bar{\Psi}_3 + (4\bar{\beta} - \bar{\tau})\bar{\Psi}_4.
\end{aligned} \tag{14}$$

The $\lambda_i^a \equiv (l^a, n^a, m^a, \bar{m}^a)$ are the components of a null tetrad, the Ψ 's and $\bar{\Psi}$'s are the components of the Weyl tensor, $\lambda_i^a \nabla_a = (D, \Delta, \delta, \bar{\delta})$ are the directional derivatives. All the other variables are the spin-coefficients. The Einstein equations are already built into the system by virtue of the Ricci tensor having been set to zero.

3 III. Left-Flat Spin-Coefficient Version of Einstein equations

There are now several different conditions that are now imposed on these equations to restrict them to left flat equations and in addition to greatly simplify them.

The first - and most basic - is to impose the left-flat (or self-dual) condition on the equations. This is accomplished by simply taking

$$(\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4) \equiv 0, \tag{15}$$

which is equivalent to imposing on the Weyl tensor C_{abcd} that

$$C_{abcd}\epsilon^{cdef} = iC_{ab}{}^{\cdot\cdot ef}. \tag{16}$$

All the barred variables (e.g., $\bar{\sigma}, \bar{\lambda}, \dots$) are independent and freed from their dual counterparts. This is denoted by replacing the bars by tildes, (e.g., $\tilde{\sigma}, \tilde{\lambda}, \dots$).

The other conditions (both coordinate and tetrad conditions) are imposed on the tetrad vectors, λ_i^a . We choose a world-line, l , in the complex thickened R^4 and a one parameter family of (complex) light cones with apex on l labeled by the complex coordinate $x^0 = u$. The null generators of the cones (null geodesics) are labeled by the points of the thickened sphere with complex stereographic coordinates $x^A = (\zeta, \tilde{\zeta})$, with $\tilde{\zeta} \approx \bar{\zeta}$. The affine parameter of the null geodesics is the complex radial coordinate $x^1 = r$. These conditions allow us¹⁸ to write the directional derivatives (or tetrad) as

$$D = l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r} \tag{17}$$

$$\nabla = n^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X^A \frac{\partial}{\partial x^A}, \quad (x^2, x^3) = (\zeta, \tilde{\zeta}), \tag{18}$$

$$\delta = m^a \frac{\partial}{\partial x^a} = \omega \frac{\partial}{\partial r} + \xi^A \frac{\partial}{\partial x^A}, \tag{19}$$

$$\tilde{\delta} = \tilde{m}^a \frac{\partial}{\partial x^a} = \tilde{\omega} \frac{\partial}{\partial r} + \tilde{\xi}^A \frac{\partial}{\partial x^A}, \tag{20}$$

and associated metric, $[g^{ab} = l^a n^b + n^a l^b - m^a \tilde{m}^b - \tilde{m}^a m^b]$,

$$g^{ab} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & g^{11} & g^{1A} \\ 0 & g^{1A} & g^{AB} \end{bmatrix} \quad (21)$$

with

$$\begin{aligned} g^{22} &= 2(U - \omega \tilde{\omega}), \\ g^{2A} &= X^A - (\tilde{\omega} \xi^A + \omega \tilde{\xi}^A), \\ g^{AB} &= -(\xi^A \tilde{\xi}^B + \tilde{\xi}^A \xi^B). \end{aligned} \quad (22)$$

By appropriate tetrad transformations¹⁸, (parallel propagation of n^a, m^a, \tilde{m}^a and appropriate scaling of l^a) we can put the following conditions on the spin-coefficients:

$$\kappa = \pi = \epsilon = \tilde{\kappa} = \tilde{\pi} = \tilde{\epsilon} = \rho - \tilde{\rho} = 0. \quad (23)$$

We also have, but do not use immediately,

$$\tau = \tilde{\alpha} + \beta, \quad \tilde{\tau} = \alpha + \tilde{\beta}. \quad (24)$$

These conditions are then inserted into our Einstein field equations to obtain our final set of field equations. The equations are displayed in groups: first, (17) of them, that contain the D derivatives, then, (7), containing only the angular derivatives, $(\delta, \tilde{\delta})$ and finally, (20), containing the time derivative, Δ .

$$D\rho = \rho^2 + \sigma\tilde{\sigma} \quad (25)$$

$$D\sigma = 2\rho\sigma, \quad (26)$$

$$D\tilde{\sigma} = 2\rho\tilde{\sigma} + \tilde{\Psi}_0 \quad (27)$$

$$D\tau = \tau\rho + \tilde{\tau}\sigma, \quad (28)$$

$$D\tilde{\tau} = \tilde{\tau}\rho + \tau\tilde{\sigma} + \tilde{\Psi}_1 \quad (29)$$

$$D\alpha = \rho\alpha + \beta\tilde{\sigma} \quad (30)$$

$$D\tilde{\alpha} = \rho\tilde{\alpha} + \tilde{\beta}\sigma \quad (31)$$

$$D\beta = \alpha\sigma + \rho\beta \quad (32)$$

$$D\tilde{\beta} = \tilde{\alpha}\tilde{\sigma} + \rho\tilde{\beta} + \tilde{\Psi}_1 \quad (33)$$

$$D\gamma = \tau\alpha + \tilde{\tau}\beta \quad (34)$$

$$D\tilde{\gamma} = \tilde{\tau}\tilde{\alpha} + \tau\tilde{\beta} + \tilde{\Psi}_2 \quad (35)$$

$$D\lambda = \rho\lambda + \tilde{\sigma}\mu \quad (36)$$

$$D\tilde{\lambda} = \rho\tilde{\lambda} + \sigma\tilde{\mu} \quad (37)$$

$$D\mu = \rho\mu + \sigma\lambda \quad (38)$$

$$D\tilde{\mu} = \rho\tilde{\mu} + \tilde{\sigma}\tilde{\lambda} + \tilde{\Psi}_2 \quad (39)$$

$$D\nu = \tilde{\tau}\mu + \tau\lambda \quad (40)$$

$$D\tilde{\nu} = \tau\tilde{\mu} + \tilde{\tau}\tilde{\lambda} + \tilde{\Psi}_3 \quad (41)$$

$$DU = \tau\tilde{\omega} + \tilde{\tau}\omega - (\gamma + \tilde{\gamma}), \quad (42)$$

$$DX^A = \tau\tilde{\xi}^A + \tilde{\tau}\xi^A, \quad (43)$$

$$D\omega = \sigma\tilde{\omega} + \rho\omega - \tau, \quad (44)$$

$$D\tilde{\omega} = \tilde{\sigma}\omega + \rho\tilde{\omega} - \tilde{\tau}, \quad (45)$$

$$D\xi^A = \sigma\tilde{\xi}^A + \rho\xi^A, \quad (46)$$

$$D\tilde{\xi}^A = \tilde{\sigma}\xi^A + \rho\tilde{\xi}^A, \quad (47)$$

$$\delta\tilde{\Psi}_0 - D\tilde{\Psi}_1 = 4\tilde{\alpha}\tilde{\Psi}_0 - 4\rho\tilde{\Psi}_1, \quad (48a)$$

$$\delta\tilde{\Psi}_1 - D\tilde{\Psi}_2 = \tilde{\lambda}\tilde{\Psi}_0 + 2\tilde{\alpha}\tilde{\Psi}_1 - 3\rho\tilde{\Psi}_2, \quad (48b)$$

$$\delta\tilde{\Psi}_2 - D\tilde{\Psi}_3 = 2\tilde{\lambda}\tilde{\Psi}_1 - 2\rho\tilde{\Psi}_3, \quad (48c)$$

$$\delta\tilde{\Psi}_3 - D\tilde{\Psi}_4 = 3\tilde{\lambda}\tilde{\Psi}_2 - 2\tilde{\alpha}\tilde{\Psi}_3 - \rho\tilde{\Psi}_4, \quad (48d)$$

$$\delta\rho - \tilde{\delta}\sigma = \rho(\tilde{\alpha} + \beta) - \sigma(3\alpha - \tilde{\beta}) \quad (49)$$

$$\tilde{\delta}\rho - \delta\tilde{\sigma} = \rho(\alpha + \tilde{\beta}) - \tilde{\sigma}(3\tilde{\alpha} - \beta) - \tilde{\Psi}_1 \quad (50)$$

$$\delta\alpha - \tilde{\delta}\beta = \mu\rho - \lambda\sigma + \alpha\tilde{\alpha} + \beta\tilde{\beta} - 2\alpha\beta \quad (51)$$

$$\tilde{\delta}\tilde{\alpha} - \delta\tilde{\beta} = \tilde{\mu}\rho - \tilde{\lambda}\tilde{\sigma} + \alpha\tilde{\alpha} + \beta\tilde{\beta} - 2\tilde{\alpha}\tilde{\beta} - \tilde{\Psi}_2 \quad (52)$$

$$\delta\lambda - \tilde{\delta}\mu = \mu(\alpha + \tilde{\beta}) + \lambda(\tilde{\alpha} - 3\beta) \quad (53)$$

$$\tilde{\delta}\tilde{\lambda} - \delta\tilde{\mu} = \tilde{\mu}(\tilde{\alpha} + \beta) + \tilde{\lambda}(\alpha - 3\tilde{\beta}) - \tilde{\Psi}_3 \quad (54)$$

$$\tilde{\delta}\omega - \delta\tilde{\omega} = (\tilde{\mu} - \mu) - (\tilde{\alpha} - \beta)\tilde{\omega} + (\alpha - \tilde{\beta})\omega. \quad (55)$$

$$\tilde{\delta}\xi^A - \delta\tilde{\xi}^A = -(\tilde{\alpha} - \beta)\tilde{\xi}^A + (\alpha - \tilde{\beta})\xi^A \quad (56)$$

$$\delta U - \Delta\omega = -\nu + \tilde{\lambda}\tilde{\omega} + (\mu - \gamma + \tilde{\gamma})\omega, \quad (57)$$

$$\tilde{\delta}U - \Delta\tilde{\omega} = -\tilde{\nu} + \lambda\omega + (\tilde{\mu} - \tilde{\gamma} + \gamma)\tilde{\omega}, \quad (58)$$

$$\delta X^A - \Delta\xi^A = \tilde{\lambda}\tilde{\xi}^A + (\mu - \gamma + \tilde{\gamma})\xi^A \quad (59)$$

$$\tilde{\delta}X^A - \Delta\tilde{\xi}^A = \lambda\xi^A + (\tilde{\mu} - \tilde{\gamma} + \gamma)\tilde{\xi}^A \quad (60)$$

$$\Delta\lambda - \tilde{\delta}\nu = -(\mu + \tilde{\mu})\lambda - (3\gamma - \tilde{\gamma}) + 2\alpha\lambda \quad (61)$$

$$\Delta\tilde{\lambda} - \delta\tilde{\nu} = -(\mu + \tilde{\mu})\tilde{\lambda} - (3\tilde{\gamma} - \gamma) + 2\tilde{\alpha}\tilde{\lambda} - \tilde{\Psi}_4 \quad (62)$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\tilde{\lambda} + \mu(\gamma + \tilde{\gamma}) - 2\beta\nu \quad (63)$$

$$\tilde{\delta}\tilde{\nu} - \Delta\tilde{\mu} = \tilde{\mu}^2 + \lambda\tilde{\lambda} + \tilde{\mu}(\gamma + \tilde{\gamma}) - 2\tilde{\beta}\tilde{\nu} \quad (64)$$

$$\delta\gamma - \Delta\beta = \mu\tau - \sigma\nu - \beta(\gamma - \tilde{\gamma} - \mu) + \alpha\tilde{\lambda} \quad (65)$$

$$\tilde{\delta}\tilde{\gamma} - \Delta\tilde{\beta} = \tilde{\mu}\tilde{\tau} - \tilde{\sigma}\tilde{\nu} - \tilde{\beta}(\tilde{\gamma} - \gamma - \tilde{\mu}) + \tilde{\alpha}\lambda \quad (66)$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \rho\tilde{\lambda}) + \tau(\tau + \beta - \tilde{\alpha}) - \sigma(3\gamma - \tilde{\gamma}) \quad (67)$$

$$\tilde{\delta}\tilde{\tau} - \Delta\tilde{\sigma} = (\tilde{\mu}\tilde{\sigma} + \rho\lambda) + \tilde{\tau}(\tilde{\tau} + \tilde{\beta} - \alpha) - \tilde{\sigma}(3\tilde{\gamma} - \gamma) \quad (68)$$

$$\Delta\rho - \tilde{\delta}\tau = -(\rho\tilde{\mu} + \sigma\lambda) + \tau(\tilde{\beta} - \alpha - \tilde{\tau}) + (\gamma + \tilde{\gamma})\rho \quad (69)$$

$$\Delta\rho - \delta\tilde{\tau} = -(\rho\mu + \tilde{\sigma}\tilde{\lambda}) + \tilde{\tau}(\beta - \tilde{\alpha} - \tau) + (\gamma + \tilde{\gamma})\rho - \tilde{\Psi}_2 \quad (70)$$

$$\Delta\alpha - \tilde{\delta}\gamma = \nu\rho - \lambda(\tau + \beta) + \alpha(\tilde{\gamma} - \tilde{\mu}) + \gamma(\tilde{\beta} - \tilde{\tau}) \quad (71)$$

$$\Delta\tilde{\alpha} - \delta\tilde{\gamma} = \tilde{\nu}\rho - \tilde{\lambda}(\tilde{\tau} + \tilde{\beta}) + \tilde{\alpha}(\gamma - \mu) + \tilde{\gamma}(\beta - \tau) - \tilde{\Psi}_3 \quad (72)$$

$$\Delta\tilde{\Psi}_0 - \tilde{\delta}\tilde{\Psi}_1 = (4\tilde{\gamma} - \tilde{\mu})\tilde{\Psi}_0 - 2(2\tilde{\tau} + \tilde{\beta})\tilde{\Psi}_1 + 3\tilde{\sigma}\tilde{\Psi}_2, \quad (73)$$

$$\Delta\tilde{\Psi}_1 - \tilde{\delta}\tilde{\Psi}_2 = \tilde{\nu}\tilde{\Psi}_0 + 2(\tilde{\gamma} - \tilde{\mu})\tilde{\Psi}_1 - 3\tilde{\tau}\tilde{\Psi}_2 + 2\tilde{\sigma}\tilde{\Psi}_3, \quad (74)$$

$$\Delta\tilde{\Psi}_2 - \tilde{\delta}\tilde{\Psi}_3 = 2\tilde{\nu}\tilde{\Psi}_1 - 3\tilde{\mu}\tilde{\Psi}_2 + 2(\tilde{\beta} - \tilde{\tau})\tilde{\Psi}_3 + \tilde{\sigma}\tilde{\Psi}_4, \quad (75)$$

$$\Delta\tilde{\Psi}_3 - \tilde{\delta}\tilde{\Psi}_4 = 3\tilde{\nu}\tilde{\Psi}_2 - 2(\tilde{\gamma} + 2\tilde{\mu})\tilde{\Psi}_3 + (4\tilde{\beta} - \tilde{\tau})\tilde{\Psi}_4. \quad (76)$$

In the following section we integrate explicitly all the D equations. The solution to each equation will have an associated 'constant' (actually a function independent of r) of integration. These 'constants' are determined by either the angular equations or by the fact of all the null surfaces being light-cones with their origin on a world-line. The evolution equations (i.e., those with Δ) turn out - initially - to be extremely complicated to deal with. Nevertheless - and quite surprising - in the end they all are equivalent to one single non-linear wave equation.

4 IV. Integration

4.1 Radial Integration

4.1.1 preliminary integration

The first thing to note is that Eqs.(25) and (26) can be integrated immediately, (with the coordinate origin for r taken at the apex of the light-cone) as

$$\rho = -\frac{1}{r}, \quad (77)$$

$$\sigma = 0. \quad (78)$$

The later follows from the fact that σ vanishes at a light-cone apex.

The field equations are greatly simplified by these results. The spin-coefficient Eqs.(28,31, 32, 37),38 decouple from the remainder and using the results from flat space light-cones with origins on a world-line¹ we have

$$\tau = \bar{\alpha} + \beta = 0, \quad (79)$$

$$\bar{\alpha} = r^{-1}\tilde{\alpha}^0 = -\frac{1}{2}r^{-1}\partial P, \quad P = 1 + \zeta\tilde{\zeta}, \quad (80)$$

$$\beta = -\tilde{\alpha}, \quad (81)$$

$$\tilde{\lambda} = 0, \quad (82)$$

$$\mu = -r^{-1}. \quad (83)$$

The form of the function P , (80), which determines the 2-sphere metric at the cones apex via Eqs.(22), is set as a coordinate condition.

In a similar fashion, with $\tau = 0$, the metric Eqs.(44), (46) integrate to

$$\xi^A = r^{-1}(\xi^{0\zeta}, \xi^{0\tilde{\zeta}}) = -r^{-1}(P, 0), \quad (84)$$

$$\omega = 0. \quad (85)$$

4.1.2 The Radial Bianchi Identities & Weyl Tensor

Turning now to our first non-trivial integrations, using the preceding results, the four radial Bianchi Identities, (48a)-48d) become

$$\partial_r \tilde{\Psi}_1 = -4r^{-1} \tilde{\Psi}_1 - r^{-1} \bar{\partial} \tilde{\Psi}_0, \quad (86a)$$

$$\partial_r \tilde{\Psi}_2 = -3r^{-1} \tilde{\Psi}_2 - r^{-1} \bar{\partial} \tilde{\Psi}_1, \quad (86b)$$

$$\partial_r \tilde{\Psi}_3 = -2r^{-1} \tilde{\Psi}_3 - r^{-1} \bar{\partial} \tilde{\Psi}_2, \quad (86c)$$

$$\partial_r \tilde{\Psi}_4 = -r^{-1} \tilde{\Psi}_4 - r^{-1} \bar{\partial} \tilde{\Psi}_3, \quad (86d)$$

where Eqs.(17) and (20), along with $\bar{\partial} \eta_s = P^{1-s} \frac{\partial}{\partial \zeta} (P^s \eta_s)$, were used.

Writing these equations, with $n = 1, 2, 3, 4$, succinctly as

$$\partial_r \tilde{\Psi}_{5-n} = -nr^{-1} \tilde{\Psi}_{5-n} - r^{-1} \bar{\partial} \tilde{\Psi}_{4-n} \quad (87)$$

the solutions are given by

$$\tilde{\Psi}_{5-n} = r^{-n} \tilde{\Psi}_{5-n}^{(0)} - r^{-n} \bar{\partial} \int^r r^{n-1} \tilde{\Psi}_{4-n} dr, \quad (88)$$

the integral being indefinite, with $\tilde{\Psi}_{5-n}^{(0)}$ the "constants of integration".

Our integration process continues with the introduction of the set of five, spin-weight $s = -2$, potentials, $\tilde{\Upsilon}_{5-n}(r)$, ($n = 1, 2, 3, 4, 5$) by

$$\tilde{\Psi}_{5-n} = (-1)^{n-1} r^{-n+2} \bar{\partial}^{(5-n)} \partial_r \tilde{\Upsilon}_{6-n} \quad (89)$$

and inserting them into the integral of Eq.(88).

Integration leads to

$$\tilde{\Psi}_{5-n} = r^{-n} \tilde{\Psi}_{5-n(0)}^{(0)} - (-1)^n r^{-n} \bar{\partial}^{(5-n)} \tilde{\Upsilon}_{5-n}(r), \quad (90)$$

where, from regularity at $r = 0$, we have

$$\tilde{\Psi}_{5-n(0)}^{(0)} = 0. \quad (91)$$

Remark 1 *We emphasize that there are two different types of situations. In one case the potentials, $\tilde{\Upsilon}$, vanish sufficiently fast at $r = 0$ so that the origin is a regular point of the manifold. In the other case there are intrinsic singularities hidden in the $\tilde{\Upsilon}(r)$ at $r = 0$. See the following section for examples.]*

Written explicitly, Eqs.(90) are

$$\tilde{\Psi}_0 = r^{-5} \tilde{\Upsilon}_0, \quad (92)$$

$$\tilde{\Psi}_1 = -r^{-4} \bar{\partial} \tilde{\Upsilon}_1, \quad (93)$$

$$\tilde{\Psi}_2 = r^{-3} \bar{\partial}^2 \tilde{\Upsilon}_2, \quad (94)$$

$$\tilde{\Psi}_3 = -r^{-2} \bar{\partial}^3 \tilde{\Upsilon}_3, \quad (95)$$

$$\tilde{\Psi}_4 = r^{-1} \bar{\partial}^4 \tilde{\Upsilon}_4. \quad (96)$$

By equating Eqs.(89) and (90), we find the ‘ladder’ relationship between the different potentials,

$$\check{\Upsilon}_{5-n} = r^2 \partial_r \check{\Upsilon}_{6-n} \quad (97)$$

or

$$\check{\Upsilon}_0 = r^2 \partial_r \check{\Upsilon}_1, \quad (98)$$

$$\check{\Upsilon}_1 = r^2 \partial_r \check{\Upsilon}_2, \quad (99)$$

$$\check{\Upsilon}_2 = r^2 \partial_r \check{\Upsilon}_3, \quad (100)$$

$$\check{\Upsilon}_3 = r^2 \partial_r \check{\Upsilon}_4. \quad (101)$$

Knowledge of $\check{\Upsilon}_4$ (which we can consider as free initial data) determines the others or equivalently, from $\check{\Upsilon}_0$ with the four constants of integration, from going up the ladder to $\check{\Upsilon}_4$, determines the others.

In the case of the regularity of the Weyl components at $r = 0$, we must impose conditions on the $\check{\Upsilon}_n$ so that near $r = 0$

$$\check{\Upsilon}_n = O(r^{5-n}). \quad (102)$$

This, with suitable differentiability, follows from $\check{\Upsilon}_4 = Ar + O(r^2)$.

For conventional asymptotic flatness (peeling) we require that, as $r \rightarrow \infty$, all the potentials tend to a constant, $\check{\Upsilon}_n = c_n + O(r^{-1})$.

From Eq.(97) we have the asymptotic behavior of the potentials:

$$\check{\Upsilon}_0 = r^2 \partial_r \check{\Upsilon}_1 = c_0 + 0(r^{-1}), \quad (103)$$

$$\check{\Upsilon}_1 = r^2 \partial_r \check{\Upsilon}_2 = c_1 - c_0 r^{-1} + 0(r^{-2}), \quad (104)$$

$$\check{\Upsilon}_2 = r^2 \partial_r \check{\Upsilon}_3 = c_2 - c_1 r^{-1} + \frac{1}{2} c_0 r^{-2} + 0(r^{-3}), \quad (105)$$

$$\check{\Upsilon}_3 = r^2 \partial_r \check{\Upsilon}_4 = c_3 - c_2 r^{-1} + \frac{1}{2} c_1 r^{-2} - \frac{1}{6} c_0 r^{-3} + 0(r^{-4}), \quad (106)$$

$$\check{\Upsilon}_4 = c_4 - c_3 r^{-1} + \frac{1}{2} c_2 r^{-2} - \frac{1}{6} c_1 r^{-3} + \frac{1}{24} c_0 r^{-4} + 0(r^{-5}). \quad (107)$$

4.1.3 Remaining Radial Equations

The remaining spin-coefficient and metric equations can be integrated and expressed in terms of the potentials.

We first investigate Eq.(27) using $\tilde{\Psi}_0 = r^{-5} \check{\Upsilon}_0$, i.e.,

$$D\tilde{\sigma} = -2r^{-1}\tilde{\sigma} + r^{-5}\check{\Upsilon}_0. \quad (108)$$

Its solution is given by

$$\tilde{\sigma} = \tilde{\sigma}^0 r^{-2} + r^{-2} \int^r r^{-3} \check{\Upsilon}_0 dr. \quad (109)$$

with (again) an indefinite integral and $\tilde{\sigma}^0$ being the constant of integration.

The integral term can be greatly simplified via repeated use of Eq.(97).
From

$$\begin{aligned}
\int^r r^{-3} \check{\Upsilon}_0 dr &= \int^r r^{-1} \partial_r \check{\Upsilon}_1 dr = \int^r [\partial_r (r^{-1} \check{\Upsilon}_1) + r^{-2} \check{\Upsilon}_1] dr \quad (110) \\
&= r^{-1} \check{\Upsilon}_1 + \int^r r^{-2} \check{\Upsilon}_1 dr = r^{-1} \check{\Upsilon}_1 + \int^r \partial_r \check{\Upsilon}_2 dr \\
&= r^{-1} \check{\Upsilon}_1 + \check{\Upsilon}_2
\end{aligned}$$

we have

$$\tilde{\sigma} = \tilde{\sigma}^0 r^{-2} + r^{-3} \check{\Upsilon}_1 + r^{-2} \check{\Upsilon}_2 \quad (111)$$

From the vanishing of the $\check{\Upsilon}$ at $r = 0$ and the regularity of the light-cones at the origin, we take $\tilde{\sigma}^0 = 0$, so that

$$\tilde{\sigma} = r^{-3} \check{\Upsilon}_1 + r^{-2} \check{\Upsilon}_2. \quad (112)$$

Using the same methods, i.e., the formal radial integration of the equations followed by the repeated use of the ladder relations, Eq.(97), all the remaining radial equations can be integrated and expressed in terms of the $\check{\Upsilon}_n$. Many of the associated "constants of integration" are determined by the behavior near $r = 0$, several are determined by use of the angular equation. (It is likely that even those determined from the angular equations could have been determined from their $r = 0$ behavior.)

The following is the full set of solutions to all the spin-coefficient and metric radial equations.

Spin-coefficients:

$$\sigma = \tau = \tilde{\alpha} + \beta = 0, \quad \tilde{\tau} = \alpha + \tilde{\beta}, \quad (113)$$

$$\rho = \tilde{\rho} = -r^{-1}, \quad (114)$$

$$\tilde{\sigma} = r^{-2}\tilde{\Upsilon}_2 + r^{-3}\tilde{\Upsilon}_1, \quad (115)$$

$$\tilde{\tau} = -r^{-2}\tilde{\sigma}\tilde{\Upsilon}_2 - r^{-1}\tilde{\sigma}\tilde{\Upsilon}_3, \quad (116)$$

$$\tilde{\alpha} = r^{-1}\tilde{\alpha}^0 = -\frac{1}{2}r^{-1}\partial_\zeta P, \quad (117)$$

$$\beta = r^{-1}\beta^0 = -\tilde{\alpha}, \quad (118)$$

$$\alpha = -r^{-1}\left(\frac{1}{2}\partial_\zeta P - \partial_\zeta P\tilde{\Upsilon}_3\right) + r^{-2}\frac{1}{2}\partial_\zeta P\tilde{\Upsilon}_2, \quad (119)$$

$$\tilde{\beta} = r^{-1}\left\{\frac{1}{2}\partial_\zeta P - \tilde{\sigma}\tilde{\Upsilon}_3 - \partial_\zeta P\tilde{\Upsilon}_3\right\} - r^{-2}\left(\tilde{\sigma}\tilde{\Upsilon}_2 + \frac{1}{2}\partial_\zeta P\tilde{\Upsilon}_2\right) \quad (120)$$

$$\gamma = -\partial_\zeta P\tilde{\sigma}\tilde{\Upsilon}_4 - r^{-1}\frac{1}{2}\partial_\zeta P\tilde{\sigma}\tilde{\Upsilon}_3 \quad (121)$$

$$\tilde{\gamma} = \tilde{\sigma}^2\tilde{\Upsilon}_4 + \partial_\zeta P\tilde{\sigma}\tilde{\Upsilon}_4 + r^{-1}(\tilde{\sigma}^2\tilde{\Upsilon}_3 + \frac{1}{2}\partial_\zeta P\tilde{\sigma}\tilde{\Upsilon}_3) \quad (122)$$

$$\tilde{\lambda} = 0 \quad (123)$$

$$\mu = -r^{-1} \quad (124)$$

$$\lambda = -2r^{-1}\tilde{\Upsilon}_3 - r^{-2}\tilde{\Upsilon}_2 \quad (125)$$

$$\tilde{\mu} = -r^{-1} + r^{-1}\tilde{\sigma}^2\tilde{\Upsilon}_3 \quad (126)$$

$$\tilde{\nu} = -\tilde{\sigma}^3\tilde{\Upsilon}_4 \quad (127)$$

$$\nu = 2\tilde{\sigma}\tilde{\Upsilon}_4 + r^{-1}\tilde{\sigma}\tilde{\Upsilon}_3 \quad (128)$$

Metric Variables:

$$\xi^A = (\xi^\zeta, \xi^{\tilde{\zeta}}) = r^{-1}\xi^{0A} = -r^{-1}(P, 0), \quad P = 1 + \zeta\tilde{\zeta} \quad (129)$$

$$\tilde{\xi}^A = (\tilde{\xi}^\zeta, \tilde{\xi}^{\tilde{\zeta}}) = -r^{-1}(0, P) - (P, 0)(2r^{-1}\tilde{\Upsilon}_3 + r^{-2}\tilde{\Upsilon}_2) \quad (130)$$

$$\omega = r^{-1}\omega^0 = 0 \quad (131)$$

$$\tilde{\omega} = r^{-1}\tilde{\omega}^0 + \tilde{\sigma}\tilde{\Upsilon}_3 = \tilde{\sigma}\tilde{\Upsilon}_3 \quad (132)$$

$$X^A = (X^\zeta, X^{\tilde{\zeta}}) = (P, 0)[r^{-1}\tilde{\sigma}\tilde{\Upsilon}_3 + 2\tilde{\sigma}\tilde{\Upsilon}_4] \quad (133)$$

$$U = -1 - r\tilde{\sigma}^2\tilde{\Upsilon}_4 \quad (134)$$

From these results we have the derivative operators which are needed in the angular and evolution equations, namely

$$D = \partial_r, \quad (135)$$

$$\nabla = \partial_u - (1 + r\tilde{\sigma}^2\tilde{\Upsilon}_4)\partial_r + (r^{-1}P\tilde{\sigma}\tilde{\Upsilon}_3 + 2P\tilde{\sigma}\tilde{\Upsilon}_4)\partial_\zeta, \quad (136)$$

$$\delta = -r^{-1}P\partial_\zeta, \quad (137)$$

$$\bar{\delta} = \tilde{\sigma}\tilde{\Upsilon}_3\partial_r - r^{-1}P\partial_{\tilde{\zeta}} - P(r^{-2}\tilde{\Upsilon}_2 + 2r^{-1}\tilde{\Upsilon}_2)\partial_\zeta. \quad (138)$$

4.2 Angular Equations

Virtually all the angular derivative spin-coefficient equations turn out to be identities when the results of the previous section are used. Some determine the "constants of integration" but they are almost certainly also determined by the behavior near $r = 0$. Several examples, (already used in the previous subsection) are:

$$\delta X^A - \Delta \xi^A = \xi^A(\mu + \tilde{\gamma} - \gamma) + \tilde{\lambda} \tilde{\xi}^A \Rightarrow \gamma^0 = 0, \quad (139)$$

$$\delta \tilde{\xi}^A - \tilde{\delta} \xi^A = (\tilde{\beta} - \alpha) \xi^A + (\tilde{\alpha} - \beta) \tilde{\xi}^A \Rightarrow \tilde{\alpha}^0 = -\frac{1}{2} \partial_\zeta P, \quad (140)$$

$$\delta \tilde{\omega} - \tilde{\delta} \omega = (\tilde{\beta} - \alpha) \omega + (\tilde{\alpha} - \beta) \tilde{\omega} + (\mu - \tilde{\mu}) \Rightarrow \text{Identity}, \quad (141)$$

$$\delta U - \Delta \omega = (\mu + \tilde{\gamma} - \gamma) \omega + \tilde{\lambda} \tilde{\omega} - \tilde{\nu} \Rightarrow \tilde{\nu}^0 = 0. \quad (142)$$

4.3 Evolution Equations

Analyzing the twenty evolution equations for the five potentials, Eqs.(57)-(76), initially presented a very difficult challenge. The process of inserting all the previous results, i.e., Eqs.(92)-(96) and Eqs.(113)-(134), into the evolution equations was a daunting task. Analyzing any one could take several days. The algebra was long and complicated with the easy production of many errors. The saving aspect of it was the fact that the equations were interrelated, in the sense that final forms of many of the equations could be compared or reduced to others, thereby giving a method of checking consistency and thus finding the errors.

Towards the end of the analysis all the evolution equations depended on just the four evolutionary Bianchi Identities, Eqs.(57) - (60). They however, using Eqs.(98) - (101), are further reduced to a single equation for the last of the potentials, i.e., $\tilde{\Upsilon}_4$.

Taking the four evolutionary Bianchi Identities, Eqs.(73)-(76), substituting the results of the radial integrations - with lengthy calculations using Eqs.(98) -(101), - we obtain the following results:

$$\begin{aligned} & \tilde{\Upsilon}'_0 - \tilde{\delta} \tilde{\delta} \tilde{\Upsilon}_1 - 2 \tilde{\Upsilon}_3 \tilde{\delta}^2 \tilde{\Upsilon}_1 + 2 \tilde{\delta} \tilde{\Upsilon}_3 \tilde{\delta} \tilde{\Upsilon}_1 - 3 \tilde{\Upsilon}_2 \tilde{\delta}^2 \tilde{\Upsilon}_2 + 2 \tilde{\delta} \tilde{\Upsilon}_4 \tilde{\delta} \tilde{\Upsilon}_0 + \tilde{\delta}^2 \tilde{\Upsilon}_4 \tilde{\Upsilon}_0 - \frac{\partial}{\partial r} \tilde{\Upsilon}_0 \\ & - r^{-1} (3 \tilde{\Upsilon}_0 \tilde{\delta}^2 \tilde{\Upsilon}_3 - 2 \tilde{\delta} \tilde{\Upsilon}_3 \tilde{\delta} \tilde{\Upsilon}_0 - 4 \tilde{\Upsilon}_0 + \tilde{\Upsilon}_2 \tilde{\delta}^2 \tilde{\Upsilon}_1 - 6 \tilde{\delta} \tilde{\Upsilon}_2 \tilde{\delta} \tilde{\Upsilon}_1 + 3 \tilde{\delta}^2 \tilde{\Upsilon}_2 \tilde{\Upsilon}_1) \\ & : - r \tilde{\delta}^2 \tilde{\Upsilon}_4 \frac{\partial}{\partial r} \tilde{\Upsilon}_0 = 0, \end{aligned} \quad (143)$$

$$\begin{aligned} & - \tilde{\delta} \tilde{\Upsilon}'_1 + \tilde{\delta} \tilde{\delta}^2 \tilde{\Upsilon}_2 - 2 \tilde{\delta}^2 \tilde{\Upsilon}_4 \tilde{\delta} \tilde{\Upsilon}_1 - 2 \tilde{\delta} \tilde{\Upsilon}_4 \tilde{\delta}^2 \tilde{\Upsilon}_1 + 2 \tilde{\Upsilon}_3 \tilde{\delta}^3 \tilde{\Upsilon}_2 + 2 \tilde{\Upsilon}_2 \tilde{\delta}^3 \tilde{\Upsilon}_3 \\ & + r^{-1} (-2 \tilde{\delta} \tilde{\Upsilon}_1 - 3 \tilde{\delta} \tilde{\Upsilon}_2 \tilde{\delta}^2 \tilde{\Upsilon}_2 + \tilde{\Upsilon}_2 \tilde{\delta}^3 \tilde{\Upsilon}_2 + 2 \tilde{\Upsilon}_1 \tilde{\delta}^3 \tilde{\Upsilon}_3 - 2 \tilde{\delta}^2 \tilde{\Upsilon}_1 \tilde{\delta} \tilde{\Upsilon}_3 + \tilde{\delta}^2 \tilde{\Upsilon}_4 \tilde{\delta} \tilde{\Upsilon}_0 + \tilde{\delta}^3 \tilde{\Upsilon}_4 \tilde{\Upsilon}_0) \\ & : + r^{-2} \tilde{\delta} \tilde{\Upsilon}_0 = 0, \end{aligned} \quad (144)$$

$$\begin{aligned}
& \partial^2 \tilde{\Upsilon}'_2 - \tilde{\partial} \partial^3 \tilde{\Upsilon}_3 - 2\partial \tilde{\Upsilon}_3 \partial^3 \tilde{\Upsilon}_3 - 2\tilde{\Upsilon}_3 \partial^4 \tilde{\Upsilon}_3 - \tilde{\Upsilon}_2 \partial^4 \tilde{\Upsilon}_4 + 2\partial \tilde{\Upsilon}_4 \partial^3 \tilde{\Upsilon}_2 + 3\partial^2 \tilde{\Upsilon}_4 \partial^2 \tilde{\Upsilon}_2 \\
& + r^{-1} (2\partial^3 \tilde{\Upsilon}_2 \partial \tilde{\Upsilon}_3 + 3\partial^2 \tilde{\Upsilon}_3 \partial^2 \tilde{\Upsilon}_2 - \tilde{\Upsilon}_2 \partial^4 \tilde{\Upsilon}_3 - \partial^4 \tilde{\Upsilon}_4 \tilde{\Upsilon}_1 - 2\partial^3 \tilde{\Upsilon}_4 \partial \tilde{\Upsilon}_1 - \partial^2 \tilde{\Upsilon}_4 \partial^2 \tilde{\Upsilon}_1) \\
& : -r^{-2} \partial^2 \tilde{\Upsilon}_1 = 0,
\end{aligned} \tag{145}$$

$$\begin{aligned}
& -\partial^3 \tilde{\Upsilon}'_3 + \tilde{\partial} \partial^4 \tilde{\Upsilon}_4 - 2\partial \tilde{\Upsilon}_4 \partial^4 \tilde{\Upsilon}_3 - 4\partial^2 \tilde{\Upsilon}_4 \partial^3 \tilde{\Upsilon}_3 + 4\partial^4 \tilde{\Upsilon}_4 \partial \tilde{\Upsilon}_3 + 2\partial^5 \tilde{\Upsilon}_4 \tilde{\Upsilon}_3 \\
& : + r^{-1} (2\partial^3 \tilde{\Upsilon}_3 - 6\partial^2 \tilde{\Upsilon}_3 \partial^3 \tilde{\Upsilon}_3 + \partial^3 \tilde{\Upsilon}_2 \partial^2 \tilde{\Upsilon}_4 + 3\partial^2 \tilde{\Upsilon}_2 \partial^3 \tilde{\Upsilon}_4 + 3\partial \tilde{\Upsilon}_2 \partial^4 \tilde{\Upsilon}_4 + \tilde{\Upsilon}_2 \partial^5 \tilde{\Upsilon}_4 - 2\partial \tilde{\Upsilon}_3 \partial^4 \tilde{\Upsilon}_3) \\
& : + r^{-2} \partial^3 \tilde{\Upsilon}_2 = 0.
\end{aligned} \tag{146}$$

These equations have the following relationship to each other: the application of the operator $r^2 \partial_r$ to the lower of each of the three consecutive pairs, [using Eqs.(98) -(101)], leads to the equality with the operator $\tilde{\partial}$ applied to the upper member of the pair, e.g., $r^2 \partial_r$ applied to Eq.(146) equals $\tilde{\partial}$ applied to Eq.(145).

Furthermore these four evolutionary Bianchi Identities are closely related to all the other evolution equations by similar identities. For example, from Eq.(58), i.e.,

$$\tilde{\delta} U - \Delta \tilde{\omega} = (\tilde{\mu} + \gamma - \tilde{\gamma}) \tilde{\omega} - \nu, \tag{147}$$

we have

$$\begin{aligned}
& -\partial \tilde{\Upsilon}'_3 + \tilde{\partial} \tilde{\partial} \tilde{\Upsilon}_4 + 2\tilde{\Upsilon}_3 \partial^3 \tilde{\Upsilon}_4 - 2\partial \tilde{\Upsilon}_4 \partial^2 \tilde{\Upsilon}_3 \\
& : + r^{-1} (2\partial \tilde{\Upsilon}_3 - 2\partial \tilde{\Upsilon}_3 \partial^2 \tilde{\Upsilon}_3 + \tilde{\Upsilon}_2 \partial^3 \tilde{\Upsilon}_4 + \partial \tilde{\Upsilon}_2 \partial^2 \tilde{\Upsilon}_4) + r^{-2} \partial \tilde{\Upsilon}_2 = 0.
\end{aligned} \tag{148}$$

The application of $\tilde{\partial}$ to (148) yields Eq.(64),

$$\tilde{\delta} \tilde{\nu} - \Delta \tilde{\mu} = \tilde{\mu}^2 + \tilde{\mu}(\gamma + \bar{\gamma}) - 2\tilde{\nu} \tilde{\beta}, \tag{149}$$

or

$$\begin{aligned}
& -\partial^2 \tilde{\Upsilon}'_3 + \partial^2 \tilde{\partial} \tilde{\Upsilon}_4 + 2\tilde{\Upsilon}_3 \partial^4 \tilde{\Upsilon}_4 + 2\partial^2 \tilde{\Upsilon}_4 - 2\partial^2 \tilde{\Upsilon}_4 \partial^2 \tilde{\Upsilon}_3 - 2\partial \tilde{\Upsilon}_4 \partial^3 \tilde{\Upsilon}_3 + 2\partial \tilde{\Upsilon}_3 \partial^3 \tilde{\Upsilon}_4 \\
& + r^{-1} (2\partial^2 \tilde{\Upsilon}_3 - 2\partial \tilde{\Upsilon}_3 \partial^3 \tilde{\Upsilon}_3 - 2\partial^2 \tilde{\Upsilon}_3 \partial^2 \tilde{\Upsilon}_3 + 2\partial \tilde{\Upsilon}_2 \partial^3 \tilde{\Upsilon}_4 + \tilde{\Upsilon}_2 \partial^4 \tilde{\Upsilon}_4 + \partial^2 \tilde{\Upsilon}_2 \partial^2 \tilde{\Upsilon}_4) \\
& : = -r^{-2} \partial^2 \tilde{\Upsilon}_2.
\end{aligned} \tag{150}$$

Finally, in turn, the application of $\tilde{\partial}$ to (150) yields the Bianchi Identity (146). The commutator on a spin-weight- s function W_s , i.e.,

$$\tilde{\partial} \tilde{\partial} W_s - \tilde{\partial} \partial W_s = -2s W_s \tag{151}$$

has been used several times.

Returning to the evolutionary Bianchi Identities - leaving the first one, (143), unchanged - we can see that the last three are simplified by performing the angular integrations. They can each be rewritten as

$$\partial B_1 = 0, \quad (152)$$

$$\partial^2 B_2 = 0, \quad (153)$$

$$\partial^3 B_3 = 0, \quad (154)$$

with

$$\begin{aligned} B_1 \equiv & -\tilde{\Upsilon}'_1 + \tilde{\partial}\tilde{\partial}\tilde{\Upsilon}_2 - 2\tilde{\Upsilon}_2 - 2\tilde{\partial}\tilde{\Upsilon}_4\tilde{\partial}\tilde{\Upsilon}_1 + 2\tilde{\Upsilon}_3\tilde{\partial}^2\tilde{\Upsilon}_2 - 2\tilde{\partial}\tilde{\Upsilon}_3\tilde{\partial}\tilde{\Upsilon}_2 + 2\tilde{\partial}^2\tilde{\Upsilon}_3\tilde{\Upsilon}_2 \\ & + r^{-1}(\tilde{\Upsilon}_2\tilde{\partial}^2\tilde{\Upsilon}_2 - 2\tilde{\Upsilon}_1 - 2\tilde{\partial}\tilde{\Upsilon}_2\tilde{\partial}\tilde{\Upsilon}_2 + 2\tilde{\Upsilon}_1\tilde{\partial}^2\tilde{\Upsilon}_3 - 2\tilde{\partial}\tilde{\Upsilon}_1\tilde{\partial}\tilde{\Upsilon}_3 + \tilde{\Upsilon}_0\tilde{\partial}^2\tilde{\Upsilon}_4) + r^{-2}\tilde{\Upsilon}_0, \end{aligned} \quad (155)$$

$$\begin{aligned} B_2 \equiv & -\tilde{\Upsilon}'_2 + \tilde{\partial}\tilde{\partial}\tilde{\Upsilon}_3 - 2\tilde{\Upsilon}_3 + \tilde{\Upsilon}_2\tilde{\partial}^2\tilde{\Upsilon}_4 - 2\tilde{\partial}\tilde{\Upsilon}_4\tilde{\partial}\tilde{\Upsilon}_2 - \tilde{\partial}\tilde{\Upsilon}_3\tilde{\partial}\tilde{\Upsilon}_3 + 2\tilde{\Upsilon}_3\tilde{\partial}^2\tilde{\Upsilon}_3 \\ & + r^{-1}(\tilde{\Upsilon}_1\tilde{\partial}^2\tilde{\Upsilon}_4 - 2\tilde{\partial}\tilde{\Upsilon}_3\tilde{\partial}\tilde{\Upsilon}_2 + \tilde{\Upsilon}_2\tilde{\partial}^2\tilde{\Upsilon}_3) + r^{-2}\tilde{\Upsilon}_1, \end{aligned} \quad (156)$$

$$B_3 \equiv -\tilde{\Upsilon}'_3 + \tilde{\partial}\tilde{\partial}\tilde{\Upsilon}_4 + 2\tilde{\Upsilon}_3\tilde{\partial}^2\tilde{\Upsilon}_4 - 2\tilde{\partial}\tilde{\Upsilon}_4\tilde{\partial}\tilde{\Upsilon}_3 + r^{-1}(2\tilde{\Upsilon}_3 - \tilde{\partial}\tilde{\Upsilon}_3\tilde{\partial}\tilde{\Upsilon}_3 + \tilde{\Upsilon}_2\tilde{\partial}^2\tilde{\Upsilon}_4) + r^{-2}\tilde{\Upsilon}_2, \quad (157)$$

so that they immediately integrate to

$$B_1 = K_1, \quad (158)$$

$$B_2 = K_2, \quad (159)$$

$$B_3 = K_3, \quad (160)$$

with (K_1, K_2, K_3) , the kernels of the operators $(\partial, \partial^2, \partial^3)$, i.e., $(\partial K_1, \partial^2 K_2, \partial^3 K_3) = 0$.

The kernels, using Eq.(97), are given by

$$K_3 = A(\tilde{\zeta}) + B(\tilde{\zeta})P + C(\tilde{\zeta})P^2 + r^{-1}(D(\tilde{\zeta}) + E(\tilde{\zeta})P) + r^{-2}F(\tilde{\zeta}), \quad (161)$$

$$K_2 = -D(\tilde{\zeta}) - E(\tilde{\zeta})P - 2r^{-1}F(\tilde{\zeta}), \quad (162)$$

$$K_1 = 2F(\tilde{\zeta}). \quad (163)$$

with six arbitrary functions of $(\tilde{\zeta})$. All have angular (or wire) singularities. Since we wish to restrict ourselves to regular solutions, the K_i are set to zero here.

Our final evolution equations for the potentials then are $B_3 = B_2 = B_1 = B_0 = 0$ or

$$\begin{aligned}
& \ddot{\Upsilon}'_0 - \ddot{\partial}\ddot{\partial}\ddot{\Upsilon}_1 - 2\ddot{\Upsilon}_3\ddot{\partial}^2\ddot{\Upsilon}_1 + 2\ddot{\partial}\ddot{\Upsilon}_3\ddot{\partial}\ddot{\Upsilon}_1 - 3\ddot{\Upsilon}_2\ddot{\partial}^2\ddot{\Upsilon}_2 + 2\ddot{\partial}\ddot{\Upsilon}_4\ddot{\partial}\ddot{\Upsilon}_0 + \ddot{\partial}^2\ddot{\Upsilon}_4\ddot{\Upsilon}_0 \\
& - r^{-1}(3\ddot{\Upsilon}_0\ddot{\partial}^2\ddot{\Upsilon}_3 - 2\ddot{\partial}\ddot{\Upsilon}_3\ddot{\partial}\ddot{\Upsilon}_0 - 4\ddot{\Upsilon}_0 + \ddot{\Upsilon}_2\ddot{\partial}^2\ddot{\Upsilon}_1 - 6\ddot{\partial}\ddot{\Upsilon}_2\ddot{\partial}\ddot{\Upsilon}_1 + 3\ddot{\partial}^2\ddot{\Upsilon}_2\ddot{\Upsilon}_1) \\
& : -\frac{\partial}{\partial r}\ddot{\Upsilon}_0 - r\ddot{\partial}^2\ddot{\Upsilon}_4\frac{\partial}{\partial r}\ddot{\Upsilon}_0 = 0, \\
& \hspace{20em} (164)
\end{aligned}$$

$$\begin{aligned}
& -\ddot{\Upsilon}'_1 + \ddot{\partial}\ddot{\partial}\ddot{\Upsilon}_2 - 2\ddot{\Upsilon}_2 - 2\ddot{\partial}\ddot{\Upsilon}_4\ddot{\partial}\ddot{\Upsilon}_1 + 2\ddot{\Upsilon}_3\ddot{\partial}^2\ddot{\Upsilon}_2 - 2\ddot{\partial}\ddot{\Upsilon}_3\ddot{\partial}\ddot{\Upsilon}_2 + 2\ddot{\partial}^2\ddot{\Upsilon}_3\ddot{\Upsilon}_2 \\
& + r^{-1}(-2\ddot{\Upsilon}_1 + \ddot{\Upsilon}_2\ddot{\partial}^2\ddot{\Upsilon}_2 - 2\ddot{\partial}\ddot{\Upsilon}_2\ddot{\partial}\ddot{\Upsilon}_2 + 2\ddot{\Upsilon}_1\ddot{\partial}^2\ddot{\Upsilon}_3 - 2\ddot{\partial}\ddot{\Upsilon}_1\ddot{\partial}\ddot{\Upsilon}_3 + \ddot{\Upsilon}_0\ddot{\partial}^2\ddot{\Upsilon}_4) + r^{-2}\ddot{\Upsilon}_0 = 0, \\
& \hspace{20em} (165)
\end{aligned}$$

$$\begin{aligned}
& -\ddot{\Upsilon}'_2 + \ddot{\partial}\ddot{\partial}\ddot{\Upsilon}_3 - 2\ddot{\Upsilon}_3 + \ddot{\Upsilon}_2\ddot{\partial}^2\ddot{\Upsilon}_4 - 2\ddot{\partial}\ddot{\Upsilon}_4\ddot{\partial}\ddot{\Upsilon}_2 - \ddot{\partial}\ddot{\Upsilon}_3\ddot{\partial}\ddot{\Upsilon}_3 + 2\ddot{\Upsilon}_3\ddot{\partial}^2\ddot{\Upsilon}_3 \\
& + r^{-1}(\ddot{\Upsilon}_1\ddot{\partial}^2\ddot{\Upsilon}_4 - 2\ddot{\partial}\ddot{\Upsilon}_3\ddot{\partial}\ddot{\Upsilon}_2 + \ddot{\Upsilon}_2\ddot{\partial}^2\ddot{\Upsilon}_3) + r^{-2}\ddot{\Upsilon}_1 = 0, \\
& \hspace{20em} (166)
\end{aligned}$$

$$\begin{aligned}
& -\ddot{\Upsilon}'_3 + \ddot{\partial}\ddot{\partial}\ddot{\Upsilon}_4 + 2\ddot{\Upsilon}_3\ddot{\partial}^2\ddot{\Upsilon}_4 - 2\ddot{\partial}\ddot{\Upsilon}_4\ddot{\partial}\ddot{\Upsilon}_3 + r^{-1}(2\ddot{\Upsilon}_3 - \ddot{\partial}\ddot{\Upsilon}_3\ddot{\partial}\ddot{\Upsilon}_3 + \ddot{\Upsilon}_2\ddot{\partial}^2\ddot{\Upsilon}_4) + r^{-2}\ddot{\Upsilon}_2 = 0. \\
& \hspace{20em} (167)
\end{aligned}$$

Finally, using Eqs.(100) and (101) of the ladder relations in Eq.(167), we obtain our non-linear wave equation $\ddot{\Upsilon}_4$ which carries all the information of the evolution of the individual $\ddot{\Upsilon}_n$;

$$\begin{aligned}
& \partial_r\ddot{\Upsilon}'_4 - r^{-2}\ddot{\partial}\ddot{\partial}\ddot{\Upsilon}_4 - \partial_r^2\ddot{\Upsilon}_4 - 4r^{-1}\partial_r\ddot{\Upsilon}_4 \\
& = -2\ddot{\partial}\ddot{\Upsilon}_4\partial_r\ddot{\partial}\ddot{\Upsilon}_4 + 4\partial_r\ddot{\Upsilon}_4\ddot{\partial}^2\ddot{\Upsilon}_4 - r\partial_r\ddot{\partial}\ddot{\Upsilon}_4\partial_r\ddot{\partial}\ddot{\Upsilon}_4 + r\partial_r^2\ddot{\Upsilon}_4\ddot{\partial}^2\ddot{\Upsilon}_4. \\
& \hspace{20em} (168)
\end{aligned}$$

Since, with the ladder relations, Eq.(168) is equivalent to the previous four equations, one could use either for integration.

5 Examples

A small set of solutions can be found by make the starting ansatz

$$\ddot{\Upsilon}_4 = c_4 - c_3r^{-1} + \frac{1}{2}c_2r^{-2} - \frac{1}{6}c_1r^{-3} + \frac{1}{24}c_0r^{-4}. \quad (169)$$

This leads immediately to

$$\begin{aligned}
\check{\Upsilon}_0 &= c_0, \\
\check{\Upsilon}_1 &= c_1 - c_0 r^{-1}, \\
\check{\Upsilon}_2 &= c_2 - c_1 r^{-1} + \frac{1}{2} c_0 r^{-2}, \\
\check{\Upsilon}_3 &= c_3 - c_2 r^{-1} + \frac{1}{2} c_1 r^{-2} - \frac{1}{6} c_0 r^{-3},
\end{aligned} \tag{170}$$

the c 's being independent of r . Inserting $\check{\Upsilon}_4$ into (167), and equating powers of r^{-1} , yields the evolution equations for the individual c 's:

$$\begin{aligned}
c'_0 - \check{\partial}\check{\partial}c_1 - 2c_3\check{\partial}^2c_1 + 2\check{\partial}c_3\check{\partial}c_1 - 3c_2\check{\partial}^2c_2 + 2\check{\partial}c_4\check{\partial}c_0 + c_0\check{\partial}^2c_4 &= 0, \\
-c'_1 + \check{\partial}\check{\partial}c_2 - 2c_2 - 2\check{\partial}c_4\check{\partial}c_1 + 2c_3\check{\partial}^2c_2 - 2\check{\partial}c_3\check{\partial}c_2 + 2c_2\check{\partial}^2c_3 &= 0, \\
-c'_2 + \check{\partial}\check{\partial}c_3 - 2c_3 + c_2\check{\partial}^2c_4 - 2\check{\partial}c_4\check{\partial}c_2 - \check{\partial}c_3\check{\partial}c_3 + 2c_3\check{\partial}^2c_3 &= 0, \\
-c'_3 + \check{\partial}\check{\partial}c_4 + 2c_3\check{\partial}^2c_4 - 2\check{\partial}c_4\check{\partial}c_3 &= 0.
\end{aligned} \tag{171}$$

We obtain a system easily integrated if we make the further ansatz,

$$c_4 = 0. \tag{172}$$

The (171) reduce to

$$\begin{aligned}
c'_0 &= \check{\partial}\check{\partial}c_1 + 2c_3\check{\partial}^2c_1 - 2\check{\partial}c_3\check{\partial}c_1 + 3c_2\check{\partial}^2c_2, \\
c'_1 &= \check{\partial}\check{\partial}c_2 - 2c_2 + 2c_3\check{\partial}^2c_2 - 2\check{\partial}c_3\check{\partial}c_2 + 2c_2\check{\partial}^2c_3, \\
c'_2 &= \check{\partial}\check{\partial}c_3 - 2c_3 - \check{\partial}c_3\check{\partial}c_3 + 2c_3\check{\partial}^2c_3, \\
c'_3 &= 0.
\end{aligned} \tag{173}$$

leading to

$$\begin{aligned}
c_3 &= c_3^0(\zeta, \tilde{\zeta}), \\
c_2 &= c_2^0(\zeta, \tilde{\zeta}) + u(\check{\partial}\check{\partial}c_3^0 - 2c_3^0 - \check{\partial}c_3^0\check{\partial}c_3^0 + 2c_3^0\check{\partial}^2c_3^0) \\
c_1 &= c_1^0(\zeta, \tilde{\zeta}) + \int^u du(\check{\partial}\check{\partial}c_2 - 2c_2 + 2c_3^0\check{\partial}^2c_2 - 2\check{\partial}c_3^0\check{\partial}c_2 + 2c_2\check{\partial}^2c_3^0), \\
c_0 &= c_0^0(\zeta, \tilde{\zeta}) + \int^u du(\check{\partial}\check{\partial}c_1 + 2c_3^0\check{\partial}^2c_1 - 2\check{\partial}c_3^0\check{\partial}c_1 + 3c_2\check{\partial}^2c_2).
\end{aligned} \tag{174}$$

These solutions have in general cubic u dependence.

The special case of $c_3 = c_2 = c_1 = 0$, leaves $c_0 = c_0^0(\zeta, \tilde{\zeta})$, a time independent solution

$$\begin{aligned}
\tilde{\Upsilon}_0 &= c_0^0, \\
\tilde{\Upsilon}_1 &= -c_0^0 r^{-1}, \\
\tilde{\Upsilon}_2 &= +\frac{1}{2}c_0^0 r^{-2}, \\
\tilde{\Upsilon}_3 &= -\frac{1}{6}c_0^0 r^{-3}.
\end{aligned} \tag{175}$$

Another class of solutions of (171) starts with the ansatz, $c_0 = c_1 = c_2 = c_3 = 0$. The last of (171), becomes

$$\tilde{\partial}\tilde{\partial}c_4 = \partial_{\tilde{\zeta}}(P^4\partial_{\zeta}(P^{-2}c_4)) = 0. \tag{176}$$

The solutions, all type N, given by

$$c_4 = P^2 G(\tilde{\zeta}) + P^2 \int P^{-4} \partial_{\zeta} F(\zeta) d\zeta, \tag{177}$$

are all singular, and have, unfortunately, wire singularities.

6 Discussion

In the interests of honesty, we describe the thought perturbations that led to the present investigation. Several months ago we were working on aspects of the complexified full Einstein equations - in particular - on asymptotic shear-free null geodesic congruences, and simply noticed that we could, with great ease, integrate the radial Bianchi Identities via the introduction of the five potentials (described in the text) with their ladder relations. Though we initially had no interest in investigating the self-dual Einstein metrics, the apparent simplicity, coming from the use of the potentials, charmed us into going for the "entire thing". Briefly it even seemed possible that we might go the entire way and be able to construct *explicit* vacuum self-dual metrics directly from the initial data. It turned out that this hope was very much dashed: first of all by the complexity of the remaining equations - they involved long and hard calculations - and more important, we ended with the unpleasant non-linear wave equation not easily solvable.

So in the end the question remains: though we have found some pretty results, what did we really accomplish. These results certainly clarify the structure of the self-dual equations - but do the results have applications or are they of interest in other investigations? They appear to potentially have use for our own investigations - but we are not certain.

There are several further issues to be mentioned.

The first is a mild mystery. The data needed for solutions to our non-linear wave equation are *two complex functions*, one the value of $\tilde{\Upsilon}_4$ on an initial null

cone u_0 , i.e., an arbitrary function of $(r, \zeta, \tilde{\zeta})$ and the second, c_4 , (the asymptotic value of $\tilde{\Upsilon}_4$) a news-like function of $(u, \zeta, \tilde{\zeta})$ that drives the evolution. On the other hand, from the original formulation of the self-dual Einstein metrics, via the so-called ‘good-cut’ equation, one needs only one complex function as the needed data, namely the asymptotic Bondi shear $\sigma^0(u_B, \zeta, \tilde{\zeta})$, with u_B the Bondi time. It is probable that by the proper counting of the ranges of the arguments it can be shown that the two sets contain the same amount of data.

A second is that there should be a geometric meaning to the potentials, $\tilde{\Upsilon}_n$. We have not attempted to investigate that.

For whatever it might be worth, and it is suggestive that it is worth something, we point out that studies¹⁸ of the self-dual Yang-Mills equations in the spin-coefficient form show that, for any gauge group, the entire set of Yang-Mills equations can be encapsulated into a single nonlinear wave equation very similar to our non-linear equation, (168). The basic variable, \mathfrak{F}_2 , is one of three *matrix* valued potentials, \mathfrak{F}_i , all connected to each other by a ladder relationship analogous to the one for GR.

The Yang-Mills wave equation¹⁹ is

$$\partial_r \mathfrak{F}'_2 - \partial_r^2 \mathfrak{F}_2 - 2r^{-1} \partial_r \mathfrak{F}_2 - r^{-2} \tilde{\partial} \tilde{\partial} \mathfrak{F}_2 = [\tilde{\partial} \mathfrak{F}_2, \partial_r \mathfrak{F}_2], \quad (178)$$

with ladder relations among the potentials

$$\mathfrak{F}_1 = r^2 \partial_r \mathfrak{F}_2, \quad (179)$$

$$\mathfrak{F}_0 = r^2 \partial_r \mathfrak{F}_1 = r^2 \partial_r (r^2 \partial_r \mathfrak{F}_2). \quad (180)$$

The tetrad components of the self-dual Yang-Mills field are given by

$$\tilde{\chi}_0 = r^{-3} \mathfrak{F}_0, \quad (181)$$

$$\tilde{\chi}_1 = r^{-2} \tilde{\partial} \mathfrak{F}_1, \quad (182)$$

$$\tilde{\chi}_2 = r^{-1} \tilde{\partial}^2 \mathfrak{F}_2, \quad (183)$$

in an almost perfect analogy with the self-dual gravitational case.

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